

A Robertson-type Uncertainty Principle and Quantum Fisher Information

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Abstract

Let A_1, \dots, A_N be complex selfadjoint matrices and let ρ be a density matrix. The Robertson uncertainty principle

$$\det \{\text{Cov}_\rho(A_h, A_j)\} \geq \det \left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\}$$

gives a bound for the quantum generalized covariance in terms of the commutators $[A_h, A_j]$. The right side matrix is antisymmetric and therefore the bound is trivial (equal to zero) in the odd case $N = 2m + 1$.

Let f be an arbitrary normalized symmetric operator monotone function and let $\langle \cdot, \cdot \rangle_{\rho, f}$ be the associated quantum Fisher information. In this paper we prove the inequality

$$\det \{\text{Cov}_\rho(A_h, A_j)\} \geq \det \left\{ \frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j] \rangle_{\rho, f} \right\}$$

that gives a non-trivial bound for any $N \in \mathbb{N}$ using the commutators $[\rho, A_h]$.

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1 Introduction

Let $(V, g(\cdot, \cdot))$ be a real inner-product vector space and suppose that $v_1, \dots, v_N \in V$. The real $N \times N$ matrix $G := \{g(v_h, v_j)\}$ is positive semidefinite and one can define $\text{Vol}^g(v_1, \dots, v_N) := \sqrt{\det\{g(v_h, v_j)\}}$. If the inner product depends on a further parameter in such a way that $g(\cdot, \cdot) = g_\rho(\cdot, \cdot)$, we write $\text{Vol}^g(v_1, \dots, v_N) = \text{Vol}_\rho^g(v_1, \dots, v_N)$.

As an example, consider a probability space $(\Omega, \mathcal{G}, \rho)$ and let $V = \mathcal{L}_{\mathbb{R}}^2(\Omega, \mathcal{G}, \rho)$ be the space of square integrable real random variables endowed with the scalar product given by the covariance $\text{Cov}_\rho(A, B) := E_\rho(AB) - E_\rho(A)E_\rho(B)$. For $A_1, \dots, A_n \in \mathcal{L}_{\mathbb{R}}^2(\Omega, \mathcal{G}, \rho)$, G is the well known covariance matrix and one has

$$\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) \geq 0. \quad (1.1)$$

The expression $\det\{\text{Cov}_\rho(A_h, A_j)\}$ is known as the *generalized variance* of the random vector (A_1, \dots, A_N) and, in general, one cannot expect a stronger inequality. For instance, when $N = 1$, (1.1) just reduces to $\text{Var}_\rho(A) \geq 0$.

In non-commutative probability the situation is quite different due to the possible non-triviality of the commutators $[A_i, A_j]$. Let $M_{n,sa} := M_{n,sa}(\mathbb{C})$ be the space of all $n \times n$ self-adjoint matrices and

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let \mathcal{D}_n^1 be the set of strictly positive density matrices (faithful states). For $A, B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ define the (symmetrized) covariance as $\text{Cov}_\rho(A, B) := 1/2[\text{Tr}(\rho AB) + \text{Tr}(\rho BA)] - \text{Tr}(\rho A) \cdot \text{Tr}(\rho B)$. If A_1, \dots, A_N are self-adjoint matrices one has

$$\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) \geq \begin{cases} 0, & N = 2m + 1, \\ \det\{-\frac{i}{2}\text{Tr}(\rho[A_h, A_j])\}^{\frac{1}{2}}, & N = 2m. \end{cases} \quad (1.2)$$

Let us call (1.2) the “standard” uncertainty principle to distinguish it from other inequalities like the “entropic” uncertainty principle and similar inequalities. Inequality (1.2) is due to Heisenberg, Kennard, Robertson and Schrödinger for $N = 2$ (see [15][17] [28] [30]). The general case is due to Robertson (see [29]). Examples of recent references where inequality (1.2) plays a role are given by [31] [32] [33] [4] [3] [16].

We are not aware of any general inequality of type (1.2) giving a bound also in the odd case $N = 2m + 1$. If one considers the case $N = 1$, it is natural to seek such an inequality in terms of the commutators $[\rho, A_i]$.

The purpose of the present paper is to prove an inequality similar to (1.2) but not trivial for any $N \in \mathbb{N}$. Let \mathcal{F}_{op} be the family of symmetric normalized operator monotone functions. To each element $f \in \mathcal{F}_{op}$ one may associate a ρ -depending scalar product $\langle \cdot, \cdot \rangle_{\rho, f}$ on the self-adjoint (traceless) matrices, which is a quantum version of the Fisher information (see [25]). Let us denote the associated volume by Vol_ρ^f . We shall prove that for any $N \in \mathbb{N}^+$ (this is one of the main differences from (1.2)) and for arbitrary self-adjoint matrices A_1, \dots, A_N one has

$$\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) \geq \left(\frac{f(0)}{2} \right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]). \quad (1.3)$$

The cases $N = 1, 2, 3$ of inequality (1.3) have been proved by the joint efforts of a number of authors in several papers: S. Luo, Q. Zhang, Z. Zhang ([20] [21] [24] [22] [23]); H. Kosaki ([18]); K. Yanagi, S. Furuichi, K. Kuriyama ([34]); F. Hansen ([14]); P. Gibilisco, D. Imparato, T. Isola ([12] [5][7]).

The scheme of the paper is as follows. In Section 2 we describe the preliminary notions of operator monotone functions, matrix means and quantum Fisher information. In Section 3 we discuss a correspondence between regular and non-regular operator monotone functions that is needed in the sequel. In Section 4 we state our main result, namely the inequality (1.3); we also state other two results concerning how the right side depends on $f \in \mathcal{F}_{op}$ and the conditions to have equality in (1.3). In Section 5 we prove the main results. In Section 6 we compare the standard uncertainty principle with inequality (1.3). In Sections 7, 8 and 9 we prove some auxiliary results.

2 Operator monotone functions, matrix means and quantum Fisher information

Let $M_n := M_n(\mathbb{C})$ (resp. $M_{n,sa} := M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices). We shall denote general matrices by X, Y, \dots while letters A, B, \dots will be used for self-adjoint matrices, endowed with the Hilbert-Schmidt scalar product $\langle A, B \rangle = \text{Tr}(A^* B)$. The adjoint of a matrix X is denoted by X^\dagger while the adjoint of a superoperator $T : (M_n, \langle \cdot, \cdot \rangle) \rightarrow (M_n, \langle \cdot, \cdot \rangle)$ is denoted by T^* . Let \mathcal{D}_n be the set of strictly positive elements of M_n and $\mathcal{D}_n^1 \subset \mathcal{D}_n$ be the set of strictly positive density matrices, namely $\mathcal{D}_n^1 = \{\rho \in M_n | \text{Tr} \rho = 1, \rho > 0\}$. If it is not otherwise specified, from now on we shall treat the case of faithful states, namely $\rho > 0$.

A function $f : (0, +\infty) \rightarrow \mathbb{R}$ is said *operator monotone (increasing)* if, for any $n \in \mathbb{N}$, and $A, B \in M_n$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. An operator monotone function is said *symmetric* if $f(x) = xf(x^{-1})$ and *normalized* if $f(1) = 1$.

Definition 2.1. \mathcal{F}_{op} is the class of functions $f : (0, +\infty) \rightarrow (0, +\infty)$ such that

- (i) $f(1) = 1$,
- (ii) $tf(t^{-1}) = f(t)$,

(iii) f is operator monotone.

Example 2.1. Examples of elements of \mathcal{F}_{op} are given by the following list

$$\begin{aligned} f_{RLD}(x) &:= \frac{2x}{x+1}, & f_{WY}(x) &:= \left(\frac{1+\sqrt{x}}{2} \right)^2, \\ f_{SLD}(x) &:= \frac{1+x}{2}, & f_{WYD(\beta)}(x) &:= \beta(1-\beta) \frac{(x-1)^2}{(x^\beta-1)(x^{1-\beta}-1)}, \quad \beta \in \left(0, \frac{1}{2}\right). \end{aligned}$$

We now report Kubo-Ando theory of matrix means (see [19]) as exposed in [27].

Definition 2.2. A mean for pairs of positive matrices is a function $m : \mathcal{D}_n \times \mathcal{D}_n \rightarrow \mathcal{D}_n$ such that

- (i) $m(A, A) = A$,
- (ii) $m(A, B) = m(B, A)$,
- (iii) $A < B \implies A < m(A, B) < B$,
- (vi) $A < A', B < B' \implies m(A, B) < m(A', B')$,
- (v) m is continuous,
- (vi) $Cm(A, B)C^* \leq m(CAC^*, CBC^*)$, for every $C \in M_n$.

Property (vi) is known as the transformer inequality. We denote by \mathcal{M}_{op} the set of matrix means. The fundamental result, due to Kubo and Ando, is the following.

Theorem 2.1. There exists a bijection between \mathcal{M}_{op} and \mathcal{F}_{op} given by the formula

$$m_f(A, B) := A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

Example 2.2. The arithmetic, geometric and harmonic (matrix) means are given respectively by

$$\begin{aligned} A \nabla B &:= \frac{1}{2}(A + B), \\ A \# B &:= A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}, \\ A ! B &:= 2(A^{-1} + B^{-1})^{-1}. \end{aligned}$$

They correspond respectively to the operator monotone functions $\frac{x+1}{2}$, \sqrt{x} , $\frac{2x}{x+1}$.

Kubo and Ando [19] proved that, among matrix means, arithmetic is the largest while harmonic is the smallest.

Corollary 2.2. For any $f \in \mathcal{F}_{op}$ and for any $x, y > 0$ one has

$$\begin{aligned} \frac{2x}{1+x} &\leq f(x) \leq \frac{1+x}{2}, \\ \frac{2xy}{x+y} &\leq m_f(x, y) \leq \frac{x+y}{2}. \end{aligned}$$

In what follows, if \mathcal{N} is a differential manifold we denote by $T_\rho \mathcal{N}$ the tangent space to \mathcal{N} at the point $\rho \in \mathcal{N}$. Recall that there exists a natural identification of $T_\rho \mathcal{D}_n^1$ with the space of self-adjoint traceless matrices; namely, for any $\rho \in \mathcal{D}_n^1$

$$T_\rho \mathcal{D}_n^1 = \{A \in M_n | A = A^*, \text{Tr}(A) = 0\}.$$

A Markov morphism is a completely positive and trace preserving operator $T : M_n \rightarrow M_m$. A monotone metric is a family of Riemannian metrics $g = \{g^n\}$ on $\{\mathcal{D}_n^1\}$, $n \in \mathbb{N}$, such that

$$g_{T(\rho)}^m(TX, TX) \leq g_\rho^n(X, X)$$

holds for every Markov morphism $T : M_n \rightarrow M_m$, for every $\rho \in \mathcal{D}_n^1$ and for every $X \in T_\rho \mathcal{D}_n^1$. Usually monotone metrics are normalized in such a way that $[A, \rho] = 0$ implies $g_\rho(A, A) = \text{Tr}(\rho^{-1} A^2)$. A

monotone metric is also said a *quantum Fisher information* (QFI) because of Chentsov uniqueness theorem for commutative monotone metrics (see [2]).

Define $L_\rho(A) := \rho A$, and $R_\rho(A) := A\rho$, and observe that they are commuting self-adjoint (positive) superoperators on $M_{n,sa}$. For any $f \in \mathcal{F}_{op}$ one can define the positive superoperator $m_f(L_\rho, R_\rho)$. Now we can state the fundamental theorem about monotone metrics.

Theorem 2.3. (see [25])

There exists a bijective correspondence between monotone metrics (quantum Fisher informations) on \mathcal{D}_n^1 and normalized symmetric operator monotone functions $f \in \mathcal{F}_{op}$. This correspondence is given by the formula

$$\langle A, B \rangle_{\rho, f} := \text{Tr}(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)).$$

The metrics associated with the functions f_β are very important in information geometry and are related to Wigner-Yanase-Dyson information (see for example [8] [9] [10] [11] [12] [5] and references therein).

3 The function \tilde{f} and its properties

For $f \in \mathcal{F}_{op}$ define $f(0) := \lim_{x \rightarrow 0} f(x)$. The condition $f(0) \neq 0$ is relevant because it is a necessary and sufficient condition for the existence of the so-called radial extension of a monotone metric to pure states (see [26]). Following [14] we say that a function $f \in \mathcal{F}_{op}$ is *regular* iff $f(0) \neq 0$. The corresponding operator mean, associated QFI, etc. are said regular too.

Definition 3.1. *We introduce the sets*

$$\mathcal{F}_{op}^r := \{f \in \mathcal{F}_{op} \mid f(0) \neq 0\}, \quad \mathcal{F}_{op}^n := \{f \in \mathcal{F}_{op} \mid f(0) = 0\}.$$

Trivially one has $\mathcal{F}_{op} = \mathcal{F}_{op}^r \dot{\cup} \mathcal{F}_{op}^n$.

Proposition 3.1. [5] *For $f \in \mathcal{F}_{op}^r$ and $x > 0$ set*

$$\tilde{f}(x) := \frac{1}{2} \left[(x+1) - (x-1)^2 \frac{f(0)}{f(x)} \right].$$

Then $\tilde{f} \in \mathcal{F}_{op}^n$.

By the very definition one has the following result (see Proposition 5.7 in [5]).

Proposition 3.2. *Let $f \in \mathcal{F}_{op}^r$. The following three conditions are equivalent:*

- 1) $\tilde{f} \leq \tilde{g}$;
- 2) $m_{\tilde{f}} \leq m_{\tilde{g}}$;
- 3) $\frac{f(0)}{f(t)} \geq \frac{g(0)}{g(t)} \quad \forall t > 0$.

Let us give some more definitions.

Definition 3.2. *Suppose that $\rho \in \mathcal{D}_n^1$ is fixed. Define $X_0 := X - \text{Tr}(\rho X)I$.*

Definition 3.3. *For $A_1, A_2 \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ define covariance and variance as*

$$\begin{aligned} \text{Cov}_\rho(A_1, A_2) &:= \frac{1}{2} [\text{Tr}(\rho A_1 A_2) + \text{Tr}(\rho A_2 A_1)] - \text{Tr}(\rho A_1) \cdot \text{Tr}(\rho A_2) = \\ &= \frac{1}{2} [\text{Tr}(\rho(A_1)_0(A_2)_0) + \text{Tr}(\rho(A_2)_0(A_1)_0)] = \text{Re}\{\text{Tr}(\rho(A_1)_0(A_2)_0)\}, \\ \text{Var}_\rho(A) &:= \text{Cov}_\rho(A, A) = \text{Tr}(\rho A^2) - \text{Tr}(\rho A)^2 = \text{Tr}(\rho A_0^2). \end{aligned}$$

Suppose, now, that $A_1, A_2 \in M_{n,sa}$, $\rho \in \mathcal{D}_n^1$ and $f \in \mathcal{F}_{op}^r$. The fundamental theorem for our present purpose is given by Proposition 6.3 in [5], which is stated as follows.

Theorem 3.3.

$$\frac{f(0)}{2} \langle i[\rho, A_1], i[\rho, A_2] \rangle_{\rho, f} = \text{Cov}_\rho(A_1, A_2) - \text{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)((A_1)_0)(A_2)_0).$$

As a consequence of the spectral theorem and of Theorem 3.3 one has the following relations.

Proposition 3.4. [5] *Let $\{\varphi_i\}$ be a complete orthonormal base composed of eigenvectors of ρ , and $\{\lambda_i\}$ the corresponding eigenvalues. To self-adjoint matrices A_1, A_2 we associate matrices $\mathcal{A}^j = A^j(\rho)$ $j = 1, 2$ whose entries are given respectively by $\mathcal{A}_{kl}^j \equiv \langle (A_j)_0 \varphi_k | \varphi_l \rangle$.*

We have the following identities.

$$\begin{aligned} \text{Cov}_\rho(A_1, A_2) &= \text{Re}\{\text{Tr}(\rho(A_1)_0(A_2)_0)\} = \frac{1}{2} \sum_{k,l} (\lambda_k + \lambda_l) \text{Re}\{\mathcal{A}_{kl}^1 \mathcal{A}_{lk}^2\} \\ \frac{f(0)}{2} \langle i[\rho, A_1], i[\rho, A_2] \rangle_{\rho, f} &= \frac{1}{2} \sum_{k,l} (\lambda_k + \lambda_l) \text{Re}\{\mathcal{A}_{kl}^1 \mathcal{A}_{lk}^2\} - \sum_{k,l} m_{\tilde{f}}(\lambda_k, \lambda_l) \text{Re}\{\mathcal{A}_{kl}^1 \mathcal{A}_{lk}^2\}. \end{aligned}$$

We also need the following result (Corollary 11.5 in [5]).

Proposition 3.5. *On pure states*

$$\text{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)((A_1)_0)(A_2)_0) = 0.$$

4 Volume theorems for quantum Fisher informations

If we have a matrix $A = \{A_{kl}\}$ we write for the determinant $\det(A) = \det\{A_{kl}\}$.

Let $(V, g(\cdot, \cdot))$ be a real inner-product vector space. By $\langle u, v \rangle$ we denote the standard scalar product for vectors $u, v \in \mathbb{R}^N$.

Proposition 4.1. *Let $v_1, \dots, v_N \in V$. The real $N \times N$ matrix $G := \{g(v_h, v_j)\}$ is positive semidefinite and therefore $\det\{g(v_h, v_j)\} \geq 0$.*

Proof. Let $x := (x_1, \dots, x_N) \in \mathbb{R}^N$. We have

$$0 \leq g\left(\sum_h x_h v_h, \sum_h x_h v_h\right) = \sum_{h,j} x_h x_j g(v_h, v_j) = \langle x, G(x) \rangle.$$

□

Motivated by the case $(V, g(\cdot, \cdot)) = (\mathbb{R}^N, \langle \cdot, \cdot \rangle)$ one can give the following definition.

Definition 4.1.

$$\text{Vol}^g(v_1, \dots, v_N) := \sqrt{\det\{g(v_h, v_j)\}}.$$

Remark 4.1.

i) Obviously,

$$\text{Vol}^g(v_1, \dots, v_N) \geq 0,$$

where the equality holds if and only if $v_1, \dots, v_N \in V$ are linearly dependent.

ii) If the inner product depends on a further parameter so that $g(\cdot, \cdot) = g_\rho(\cdot, \cdot)$, we write $\text{Vol}_\rho^g(v_1, \dots, v_N) = \text{Vol}^g(v_1, \dots, v_N)$.

iii) In the case of a probability space $(V, g_\rho(\cdot, \cdot)) = (\mathcal{L}_\mathbb{R}^2(\Omega, \mathcal{G}, \rho), \text{Cov}_\rho(\cdot, \cdot))$ the number $\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N)^2$ is known as the *generalized variance* of the random vector (A_1, \dots, A_N) .

In what follows we move to the noncommutative case. Here A_1, \dots, A_N are self-adjoint matrices, ρ is a (faithful) density matrix and $g(\cdot, \cdot) = \text{Cov}_\rho(\cdot, \cdot)$ has been defined in (3.3). By Vol_ρ^f we denote the volume associated to the quantum Fisher information $\langle \cdot, \cdot \rangle_{\rho, f}$ given by the (regular) normalized symmetric operator monotone function f .

Definition 4.2.

The function

$$I_\rho^f(A) := \frac{f(0)}{2} \text{Vol}_\rho^f(i[\rho, A]) = \frac{f(0)}{2} \langle i[\rho, A], i[\rho, A] \rangle_{\rho, f}$$

is known as the *metric adjusted skew information* or *f-information* (see [13] [5]).

Let $N \in \mathbb{N}$, $f \in \mathcal{F}_{op}^r$, $\rho \in \mathcal{D}_n^1$ and $A_1, \dots, A_N \in M_{n,sa}$ be arbitrary. We shall prove in Section 5 the following results.

Theorem 4.2.

$$\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) \geq \left(\frac{f(0)}{2} \right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]). \quad (4.1)$$

Theorem 4.3. *The above inequality is an equality if and only if A_{10}, \dots, A_{N0} are linearly dependent.*

Theorem 4.4. *Fix $N \in \mathbb{N}$, $\rho \in \mathcal{D}_n^1$ and $A_1, \dots, A_N \in M_{n,sa}$. Define for $f \in \mathcal{F}_{op}^r$*

$$V(f) := \left(\frac{f(0)}{2} \right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]).$$

Then, for any $f, g \in \mathcal{F}_{op}^r$

$$\tilde{f} \leq \tilde{g} \implies V(f) \geq V(g).$$

Remark 4.2. The inequality

$$\det\{\text{Cov}_\rho(A_h, A_j)\} \geq \det\left\{\text{Cov}_\rho(A_h, A_j) - \text{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)((A_h)_0)(A_j)_0)\right\}.$$

makes sense also for not faithful states and it is true by continuity as a consequence of Theorem 5.1.

Because of Proposition 3.5 one has (by an obvious extension of the definition) the following result.

Proposition 4.5. *If ρ is a pure state then $\forall N \in \mathbb{N}$, $\forall f \in \mathcal{F}_{op}^r$, $\forall A_1, \dots, A_N \in M_{n,sa}$ one has*

$$\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) = \left(\frac{f(0)}{2} \right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]).$$

5 Proof of the main results

Theorem 5.1.

$$\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) \geq \left(\frac{f(0)}{2} \right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]) \quad \forall N \in \mathbb{N}^+, \quad \forall f \in \mathcal{F}_{op}^r.$$

Proof. Theorem 5.1 is equivalent to the following inequality

$$\det\{\text{Cov}_\rho(A_h, A_j)\} \geq \det\left\{\frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j] \rangle_{\rho, f}\right\}.$$

If ρ and A_1, \dots, A_N are fixed set

$$F(f) := \det\{\text{Cov}_\rho(A_h, A_j)\} - \det\left\{\frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j] \rangle_{\rho, f}\right\}.$$

Because of Theorem 3.3 one has

$$F(f) = \det\{\text{Cov}_\rho(A_h, A_j)\} - \det\left\{\text{Cov}_\rho(A_h, A_j) - \text{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)((A_h)_0)(A_j)_0)\right\}.$$

Theorem 5.1 is equivalent to

$$F(f) \geq 0.$$

From Proposition 3.4, we have

$$\begin{aligned} \text{Cov}_\rho(A_h, A_j) &= \text{Re}\{\text{Tr}(\rho(A_h)_0(A_j)_0)\} = \frac{1}{2} \sum_{k,l} (\lambda_k + \lambda_l) \text{Re}\{\mathcal{A}_{kl}^h A_{lk}^j\} \\ \frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j] \rangle_{\rho, f} &= \frac{1}{2} \sum_{k,l} (\lambda_k + \lambda_l) \text{Re}\{\mathcal{A}_{kl}^h \mathcal{A}_{lk}^j\} - \sum_{k,l} m_{\tilde{f}}(\lambda_k, \lambda_l) \text{Re}\{\mathcal{A}_{kl}^h A_{lk}^j\}. \end{aligned}$$

We have

$$\begin{aligned} F(f) &:= \sum_{\sigma \in S_N} \text{sgn } \sigma \left[\prod_{j=1}^N \text{Cov}_\rho(A_j, A_{\sigma(j)}) - \prod_{j=1}^N \frac{f(0)}{2} \langle i[\rho, A_j], i[\rho, A_{\sigma(j)}] \rangle_{\rho, f} \right] \\ &:= \sum_{\sigma \in S_N} \text{sgn } \sigma \xi_\sigma, \end{aligned}$$

where

$$\xi_\sigma = \prod_{j=1}^N \sum_{k,l=1}^n \frac{\lambda_k + \lambda_l}{2} \text{Re}\{\mathcal{A}_{kl}^j \mathcal{A}_{lk}^{\sigma(j)}\} - \prod_{j=1}^N \sum_{k,l=1}^n \left[\frac{\lambda_k + \lambda_l}{2} - m_{\tilde{f}}(\lambda_k, \lambda_l) \right] \text{Re}\{\mathcal{A}_{kl}^j \mathcal{A}_{lk}^{\sigma(j)}\}.$$

From the Definition 8.2 we get (applying Proposition 7.3 to the case $X = \underline{n}$)

$$\begin{aligned} \xi_\sigma &= \prod_{j=1}^N \sum_{k,l=1}^n \frac{\lambda_k + \lambda_l}{2} \text{Re}\{\mathcal{A}_{kl}^j \mathcal{A}_{lk}^{\sigma(j)}\} - \prod_{j=1}^N \sum_{k,l=1}^n \left[\frac{\lambda_k + \lambda_l}{2} - m_{\tilde{f}}(\lambda_k, \lambda_l) \right] \text{Re}\{\mathcal{A}_{kl}^j \mathcal{A}_{lk}^{\sigma(j)}\} \\ &= \sum_{\alpha, \beta \in \mathbb{C}} \left\{ \prod_{j=1}^N \frac{\lambda_{\alpha_j} + \lambda_{\beta_j}}{2} \text{Re}\{\mathcal{A}_{\alpha_j \beta_j}^j \mathcal{A}_{\beta_j \alpha_j}^{\sigma(j)}\} - \prod_{j=1}^N \left[\frac{\lambda_{\alpha_j} + \lambda_{\beta_j}}{2} - m_{\tilde{f}}(\lambda_{\alpha_j}, \lambda_{\beta_j}) \right] \text{Re}\{\mathcal{A}_{\alpha_j \beta_j}^j \mathcal{A}_{\beta_j \alpha_j}^{\sigma(j)}\} \right\} \\ &= \sum_{\alpha, \beta \in \mathbb{C}} \left\{ \prod_{j=1}^N \frac{\lambda_{\alpha_j} + \lambda_{\beta_j}}{2} \prod_{j=1}^N \text{Re}\{\mathcal{A}_{\alpha_j \beta_j}^j \mathcal{A}_{\beta_j \alpha_j}^{\sigma(j)}\} - \prod_{j=1}^N \left[\frac{\lambda_{\alpha_j} + \lambda_{\beta_j}}{2} - m_{\tilde{f}}(\lambda_{\alpha_j}, \lambda_{\beta_j}) \right] \prod_{j=1}^N \text{Re}\{\mathcal{A}_{\alpha_j \beta_j}^j \mathcal{A}_{\beta_j \alpha_j}^{\sigma(j)}\} \right\} \\ &= \sum_{\alpha, \beta \in \mathbb{C}} \left[\prod_{j=1}^N \frac{\lambda_{\alpha_j} + \lambda_{\beta_j}}{2} - \prod_{j=1}^N \left(\frac{\lambda_{\alpha_j} + \lambda_{\beta_j}}{2} - m_{\tilde{f}}(\lambda_{\alpha_j}, \lambda_{\beta_j}) \right) \right] \prod_{j=1}^N \text{Re}\{\mathcal{A}_{\alpha_j \beta_j}^j \mathcal{A}_{\beta_j \alpha_j}^{\sigma(j)}\} \\ &= \sum_{\alpha, \beta \in \mathbb{C}} H_{\alpha, \beta} \prod_{j=1}^N \text{Re}\{\mathcal{A}_{\alpha_j \beta_j}^j \mathcal{A}_{\beta_j \alpha_j}^{\sigma(j)}\}. \end{aligned}$$

Hence, applying Proposition 7.5 to the case $G = S^N$, $X = \mathbb{C} \times \mathbb{C}$ and $r(x) := r(\alpha, \beta) := H_{\alpha, \beta}^f \det \mathcal{B}^{\alpha, \beta}$ and Proposition 8.4 we get

$$\begin{aligned} F(f) &= \sum_{\sigma \in S_N} \text{sgn } \sigma \sum_{\alpha, \beta \in \mathbb{C}} H_{\alpha, \beta}^f \prod_{i=1}^N \text{Re}\{\mathcal{A}_{\alpha_i \beta_i}^i \mathcal{A}_{\beta_i \alpha_i}^{\sigma(i)}\} \\ &= \sum_{\alpha, \beta \in \mathbb{C}} H_{\alpha, \beta}^f \sum_{\sigma \in S_N} \text{sgn } \sigma \prod_{i=1}^N \text{Re}\{\mathcal{A}_{\alpha_i \beta_i}^i \mathcal{A}_{\beta_i \alpha_i}^{\sigma(i)}\} \\ &= \sum_{\alpha, \beta \in \mathbb{C}} H_{\alpha, \beta}^f \det \mathcal{B}^{\alpha, \beta} \\ &= \frac{1}{N!} \sum_{\alpha, \beta \in \mathbb{C}} H_{\alpha, \beta}^f \sum_{\sigma \in S_N} \det \mathcal{B}^{\alpha \sigma, \beta \sigma} \\ &= \frac{1}{N!} \sum_{\alpha, \beta \in \mathbb{C}} H_{\alpha, \beta}^f K_{\alpha, \beta}. \end{aligned}$$

By Corollary 8.3, $H_{\alpha,\beta}^f$ is strictly positive; on the other hand, Lemma 9.2 ensures the nonnegativity of $K_{\alpha,\beta}$, so that we can conclude. \square

Theorem 5.2. *The inequality*

$$\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) \geq \left(\frac{f(0)}{2} \right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]) \quad \forall N \in \mathbb{N}^+, \quad \forall f \in \mathcal{F}_{op}^r.$$

is an equality if and only if $(A_1)_0, \dots, (A_N)_0$ are linearly dependent.

Proof. Since

$$\text{Cov}_\rho(A_1, A_2) = \text{Tr}(\rho(A_1)_0(A_2)_0) = \text{Cov}_\rho((A_1)_0, (A_2)_0)$$

we have that

$$\text{Cov}_\rho(A_1, A_2) = \text{Cov}_\rho((A_1)_0, (A_2)_0).$$

From this it follows

$$\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) = \text{Vol}_\rho^{\text{Cov}}((A_1)_0, \dots, (A_N)_0)$$

Therefore if $(A_1)_0, \dots, (A_N)_0$ are linearly dependent then

$$0 = \text{Vol}_\rho^{\text{Cov}}((A_1)_0, \dots, (A_N)_0) = \text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) \geq \left(\frac{f(0)}{2} \right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]) \geq 0$$

and we are done.

Conversely, suppose that $(A_1)_0, \dots, (A_N)_0$ are not linear dependent; then we want to show that $F(f) > 0$. Since for any $\alpha, \beta \in \mathcal{C}$, $H_{\alpha,\beta}$ is strictly positive and $K_{\alpha,\beta}$ is nonnegative, this is equivalent to prove that $K_{\alpha,\beta}$ is not null for some $\alpha, \beta \in \mathcal{C}$. Because of Lemma 9.2, this is, in turn, equivalent to show that $\det(C^{u(j)} A_{\alpha_j, \beta_j}^i)$ is not null for some $\alpha, \beta \in \mathcal{C}$ and $u \in \{0, 1\}^N$. This is a consequence of Corollary 9.4. \square

Theorem 5.3. *Define*

$$V(f) := \left(\frac{f(0)}{2} \right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]).$$

Then

$$\tilde{f} \leq \tilde{g} \implies V(f) \geq V(g).$$

Proof. Because of Proposition 8.1 and Proposition 8.2, one has that

$$\tilde{f} \leq \tilde{g} \implies 0 < H_{\alpha,\beta}^f \leq H_{\alpha,\beta}^g.$$

Since $K_{\alpha,\beta} \geq 0$ does not depend on f and

$$F(f) = \frac{1}{N!} \sum_{\alpha, \beta \in \mathcal{C}} H_{\alpha,\beta}^f K_{\alpha,\beta}$$

we get that

$$0 \leq F(f) \leq F(g).$$

By definition of F , we obtain the thesis. \square

6 Relation with the standard uncertainty principle

Theorem 6.1. (*Hadamard inequality*)

If $H \in M_{N,sa}$ is positive semidefinite then

$$\det(H) \leq \prod_{j=1}^N h_{jj}.$$

Theorem 6.2. Let $f \in \mathcal{F}_{op}^r$. The inequality

$$\det \left\{ \frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j] \rangle_{\rho, f} \right\} \geq \det \left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\}$$

is (in general) false for any $N = 2m$.

Proof. Let $n = N = 2m$. By the Hadamard inequality it is enough to find $A_1, \dots, A_N \in M_{N,sa}$ and a state $\rho \in \mathcal{D}_N^1$ such that

$$\prod_{j=1}^N I_\rho^f(A_j) < \det \left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\}. \quad (6.1)$$

Let $\rho := \text{diag}(\lambda_1, \dots, \lambda_N)$, where $\lambda_1 < \lambda_2 < \dots < \lambda_N$. The aim is to construct A_1, \dots, A_N that are block-diagonal matrices, each matrix consisting of exactly one non-null block equal to a 2×2 Pauli matrix.

More precisely, given $h = 2q + 1$, where $q = 0, \dots, N-1$, define the Hermitian matrices A_h and A_{h+1} such that $(A_h)_{hh+1} = i = (A_h^*)_{h+1h}$, $(A_{h+1})_{hh+1} = 1 = (A_{h+1})_{h+1h}$ and $(A_h)_{kl} = (A_{h+1})_{kl} = 0$ elsewhere.

Since the state ρ is diagonal and A_h are null diagonal matrices, $A_h \equiv \mathcal{A}^h$, where $(\mathcal{A}^h)_{kl} = \langle (A_h)_0 \phi_k, \phi_l \rangle$ is defined as in Proposition 3.4. Therefore, say, if h is odd one obtains from Proposition 3.4

$$\begin{aligned} I_\rho^f(A_h) &= \frac{1}{2} \sum_{k,l} (\lambda_k + \lambda_l) |\mathcal{A}_{kl}^h|^2 - \sum_{k,l} m_{\bar{f}}(\lambda_k, \lambda_l) |\mathcal{A}_{kl}^h|^2 \\ &= \lambda_h + \lambda_{h+1} - 2m_{\bar{f}}(\lambda_h, \lambda_{h+1}) \\ &= I_\rho^f(A_{h+1}). \end{aligned}$$

Suppose now that h is odd and $h < k$. We have

$$\begin{aligned} \text{Tr}(\rho[A_h, A_k]) &= \sum_{j,l,m} \rho_{jl} ((A_h)_{lm}(A_k)_{mj} - (A_k)_{lm}(A_h)_{mj}) \\ &= \sum_{j,m} \lambda_j ((A_h)_{jm}(A_k)_{mj} - (A_k)_{jm}(A_h)_{mj}) \\ &= \sum_{j,m} \lambda_j ((A_h)_{jm}(A_k)_{mj} - (A_k)_{jm}(A_h)_{mj}) \\ &= 2i(\lambda_h - \lambda_{h+1})\delta_k^{h+1}, \end{aligned}$$

where δ_h^{k+1} denotes the Kronecker delta function. We have that $\text{Tr}(\rho[A_h, A_k]) = -\text{Tr}(\rho[A_k, A_h])$ and therefore,

$$\left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\} = \begin{pmatrix} 0 & \lambda_1 - \lambda_2 & 0 & \dots & 0 \\ \lambda_2 - \lambda_1 & 0 & \lambda_2 - \lambda_3 & \dots & 0 \\ 0 & \lambda_3 - \lambda_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \lambda_{N-1} - \lambda_N \\ 0 & 0 & \dots & \lambda_N - \lambda_{N-1} & 0 \end{pmatrix}$$

so that

$$\det \left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\} = \prod_{h < N, h=2q+1} (\lambda_{h+1} - \lambda_h)^2$$

Finally, since for any $f \in \mathcal{F}_{op}^r$ the function $m_{\bar{f}}(\cdot, \cdot)$ is a mean, one has $\lambda_h < m_{\bar{f}}(\lambda_h, \lambda_{h+1}) < \lambda_{h+1}$. This implies, for any odd h ,

$$I_{\rho}^f(A_h) = I_{\rho}^f(A_{h+1}) = \lambda_h + \lambda_{h+1} - 2m_{\bar{f}}(\lambda_h, \lambda_{h+1}) < \lambda_{h+1} - \lambda_h,$$

so that one can get (6.1) by taking the product over all h . □

Theorem 6.3. *Let $f \in \mathcal{F}_{op}^r$. The inequality*

$$\det \left\{ \frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j] \rangle_{\rho, f} \right\} \leq \det \left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\}$$

is (in general) false for any $N = 2m$.

Proof. It suffices to find selfadjoint matrices A_1, \dots, A_N which are pairwise commuting but not commuting with a given state ρ and such that $[\rho, A_1], \dots, [\rho, A_N]$ are linearly independent.

Consider a state of the form $\rho = \text{diag}(\lambda_1, \dots, \lambda_n)$ where the eigenvalues λ_i are all distinct.

Let $A_1, \dots, A_N \in \mathcal{M}_{n,sa}(\mathbb{R})$ be N linear independent symmetric real matrices such that $(A_j)_{kk} = 0$ for any $j = 1, \dots, N$ and $k = 1, \dots, n$. Note that the linear independence of A_1, \dots, A_N implies the condition $n(n-1)/2 \geq N$.

Obviously, $[A_j, A_m] = 0$ for any $j, m = 1, \dots, N$, while a direct computation shows that

$$\begin{aligned} ([\rho, A_j])_{kl} &= \sum_{h=1}^n \rho_{kh} (A_j)_{hl} - \sum_{h=1}^n (A_j)_{kh} \rho_{hl} \\ &= (A_j)_{kl} (\lambda_k - \lambda_l) \end{aligned}$$

Observe that also $[\rho, A_1] \dots [\rho, A_N]$ are linear independent. Suppose, in fact, that there exists a vector $\alpha \in \mathbb{R}^N$ such that

$$\sum_{j=1}^N \alpha_j [\rho, A_j] \equiv 0,$$

that is, for any $k, l = 1, \dots, n$

$$0 = \sum_{j=1}^N \alpha_j ([\rho, A_j])_{kl} = (\lambda_k - \lambda_l) \sum_{j=1}^N \alpha_j (A_j)_{kl}.$$

This implies that $\sum_j \alpha_j (A_j)_{kl} = 0$, and hence $\alpha \equiv 0$, because of the linear independence of A_1, \dots, A_N . □

7 Appendix A: combinatorics

Set $\underline{n} := \{1, \dots, n\}$. Moreover define

$$\mathcal{C} := \underline{n}^{\underline{N}} = \{(x_1, \dots, x_N) : x_i \in \{1, \dots, n\}, i = 1, \dots, N\}.$$

One can prove the following result.

Proposition 7.1. *For a finite set $X \subset \mathbb{N}$ and $N \in \mathbb{N}^+$ one has*

$$\prod_{j=1}^N \sum_{k \in X} Q_j^k = \sum_{u \in X^{\underline{N}}} \prod_{j=1}^N Q_j^{u(j)}$$

For $z \in \mathbb{C}$ we shall introduce the operator

$$C^k(z) := \begin{cases} \operatorname{Re}(z) & \text{if } k = 0, \\ \operatorname{Im}(z) & \text{if } k = 1. \end{cases}$$

Taking $X = \{0, 1\}$ and $Q_j^k = C^k(z_j)C^k(w_j)$ in Proposition 7.1 we get

Corollary 7.2. *If $z_j, w_j \in \mathbb{C}$ then*

$$\prod_{j=1}^N \left(\sum_{k \in \{0,1\}} C^k(z_j)C^k(w_j) \right) = \sum_{u \in \{0,1\}^{\underline{N}}} \left(\prod_{j=1}^N C^{u(j)}(z_j)C^{u(j)}(w_j) \right).$$

With similar arguments it is possible to prove the following result.

Proposition 7.3. *For a finite set $X \subset \mathbb{N}$ and $N \in \mathbb{N}^+$ one has*

$$\prod_{j=1}^N \left(\sum_{k,l \in X} Q_{kl}^j \right) = \sum_{\alpha, \beta \in X^{\underline{N}}} \left(\prod_{j=1}^N Q_{\alpha(j)\beta(j)}^j \right)$$

The following result is obvious.

Proposition 7.4. *Let X be a finite set and $g : X \rightarrow X$ a bijection. For any function $r : X \rightarrow \mathbb{R}$ one has*

$$\sum_{x \in X} r(x) = \sum_{x \in X} r(g(x)).$$

From the above result one obtains the following

Proposition 7.5. *Let X be a finite set and let G be a group of bijections $g : X \rightarrow X$. For any function $r : X \rightarrow \mathbb{R}$ one has*

$$\sum_{x \in X} r(x) = \frac{1}{\#(G)} \sum_{x \in X} \sum_{g \in G} r(g(x)).$$

We denote by S^N the symmetric group of order N .

Example 7.1. *The set $X := \{0, 1\}^{\underline{N}}$ can be identified with the power set of \underline{N} . If $u \in \{0, 1\}^{\underline{N}}$ and $\sigma \in S_N$ the σ can be seen as a bijection $\sigma : X \rightarrow X$ defining $\sigma(u) := u \circ \sigma$.*

From the above considerations we get the following Lemma.

Lemma 7.6. *For any function $r : \{0, 1\}^{\underline{N}} \rightarrow \mathbb{R}$ and for any $\sigma \in S_N$ one has*

$$\sum_{u \in \{0,1\}^{\underline{N}}} r(u) = \sum_{u \in \{0,1\}^{\underline{N}}} r(\sigma(u)).$$

Remark 7.1. If $E = \{E_{jk}\}$ and $E(\sigma) := \{E_{\sigma(j)k}\}$ one has

$$\det(E(\sigma)) = \operatorname{sgn}(\sigma)\det(E).$$

8 Appendix B: the function H

Let $\mathbb{R}_+ := (0, +\infty)$ and $\mathbf{x} = (x_1, \dots, x_N), \mathbf{y} = (y_1, \dots, y_N) \in \mathbb{R}_+^N$.

In the sequel we need to study the following function.

Definition 8.1. For any $f \in \mathcal{F}_{op}^r$, set

$$H^f(\mathbf{x}, \mathbf{y}) := \prod_{j=1}^N \frac{x_j + y_j}{2} - \prod_{j=1}^N \left(\frac{x_j + y_j}{2} - m_{\tilde{f}}(x_j, y_j) \right)$$

Proposition 8.1. For any $\mathcal{F} \in \mathcal{F}_{op}^r$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N$,

$$H^f(\mathbf{x}, \mathbf{y}) > 0.$$

Proof. Since for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N$,

$$0 < m_{\tilde{f}}(x_j, y_j) \leq \frac{x_j + y_j}{2}, \quad j = 1, \dots, N,$$

we have

$$\prod_{j=1}^N \left(\frac{x_j + y_j}{2} - m_{\tilde{f}}(x_j, y_j) \right) < \prod_{j=1}^N \frac{x_j + y_j}{2},$$

so that we can conclude. □

Proposition 8.2.

$$\begin{aligned} \tilde{f} &\leq \tilde{g} \\ \Downarrow \\ H^f(\mathbf{x}, \mathbf{y}) &\leq H^g(\mathbf{x}, \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N. \end{aligned}$$

Proof. Since for any $x, y > 0$

$$\frac{x + y}{2} - m_{\tilde{f}}(x, y) = \frac{(x - y)^2}{2y} \cdot \frac{f(0)}{f(\frac{x}{y})}, \quad (8.1)$$

we have

$$H^f(\mathbf{x}, \mathbf{y}) := \prod_{j=1}^N \frac{x_j + y_j}{2} - \prod_{j=1}^N \left(\frac{(x_j - y_j)^2}{2y_j} \cdot \frac{f(0)}{f(\frac{x_j}{y_j})} \right).$$

Because of Proposition 3.2 we have

$$\tilde{f} \leq \tilde{g} \Rightarrow \frac{f(0)}{f(t)} \geq \frac{g(0)}{g(t)} > 0 \quad \forall t > 0;$$

hence, we obtain

$$H^f(\mathbf{x}, \mathbf{y}) \leq H^g(\mathbf{x}, \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N$$

by elementary computations. □

Corollary 8.3. For any $f \in \mathcal{F}_{op}$,

$$0 < H^{SLD}(\mathbf{x}, \mathbf{y}) \leq H^f(\mathbf{x}, \mathbf{y}) \leq \frac{1}{2^N} \prod_{j=1}^N (x_j + y_j) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N.$$

Definition 8.2. Fix $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n$. Given $\alpha, \beta \in \mathcal{C} = \underline{n}^N$, let $H_{\alpha, \beta}^f := H^f(\lambda_\alpha, \lambda_\beta)$, where $\lambda_\alpha := (\lambda_{\alpha_1}, \dots, \lambda_{\alpha_N})$, $\lambda_\beta := (\lambda_{\beta_1}, \dots, \lambda_{\beta_N})$.

Proposition 8.4. For all $\sigma \in S^N$ one has

$$H_{\alpha(\sigma), \beta(\sigma)}^f = H_{\alpha, \beta}^f.$$

Proof. Left to the reader □

9 Appendix C: the function K

In order to prove the main result of this paper, we introduce some notations. Let $\{\varphi_i\}$ be a complete orthonormal base composed of eigenvectors of ρ , and $\{\lambda_i\}$ the corresponding eigenvalues. As in Proposition 3.4 set

$$\mathcal{A}_{kl}^j := \langle (A_j)_0 \varphi_k | \varphi_l \rangle \quad j = 1, \dots, N; \quad k, l = 1, \dots, n.$$

Note that since the A_j are selfadjoint one has $\mathcal{A}_{kl}^j = \overline{\mathcal{A}_{lk}^j}$ namely

$$\operatorname{Re}(\mathcal{A}_{kl}^j) = \operatorname{Re}(\mathcal{A}_{lk}^j) \quad \operatorname{Im}(\mathcal{A}_{kl}^j) = -\operatorname{Im}(\mathcal{A}_{lk}^j)$$

Since

$$\operatorname{Re}(zw) = \operatorname{Re}(z)\operatorname{Re}(w) - \operatorname{Im}(z)\operatorname{Im}(w)$$

we have

Lemma 9.1.

$$\operatorname{Re}(\mathcal{A}_{kl}^j \mathcal{A}_{lk}^m) = \operatorname{Re}(\mathcal{A}_{kl}^j) \operatorname{Re}(\mathcal{A}_{lk}^m) + \operatorname{Im}(\mathcal{A}_{kl}^j) \operatorname{Im}(\mathcal{A}_{lk}^m)$$

If $\alpha, \beta \in \mathcal{C} = \underline{n}^N$ and $\sigma \in S^N$ we define a $N \times N$ matrix $\mathcal{B}^{\alpha\sigma, \beta\sigma}$ setting

$$(\mathcal{B}^{\alpha\sigma, \beta\sigma})_{hj} := \operatorname{Re}\{\mathcal{A}_{\alpha_{\sigma(h)}, \beta_{\sigma(h)}}^h \mathcal{A}_{\beta_{\sigma(h)}, \alpha_{\sigma(h)}}^j\}. \quad h, j = 1, \dots, N; \quad \alpha_{\sigma(h)}, \beta_{\sigma(h)} = 1, \dots, n$$

When $\sigma := I$ is the identity in S_N , we shall simply denote by $\mathcal{A}_{\alpha, \beta}$ and $\mathcal{B}^{\alpha, \beta}$ the corresponding matrices.

Definition 9.1.

$$K_{\alpha, \beta} := K_{\alpha, \beta}(\rho; A_1, \dots, A_N) := \sum_{\sigma \in S_N} \det(\mathcal{B}^{\alpha\sigma, \beta\sigma})$$

Definition 9.2. If $u \in \{0, 1\}^N$ and $\alpha, \beta \in \underline{n}^N$ we define an $N \times N$ matrix setting

$$D(u; \alpha, \beta) := \{D(u; \alpha, \beta)_{hj}\} := \{C^{u(j)} \mathcal{A}_{\alpha_j \beta_j}^h\} \quad h, j = 1, \dots, N$$

Proposition 9.2. We have

$$K_{\alpha, \beta} = \sum_{u \in \{0, 1\}^N} \det(D(u; \alpha, \beta))^2 \geq 0.$$

so that $K_{\alpha, \beta} \geq 0$.

Proof. Applying:

- i) Lemma 9.1;
 - ii) Corollary 7.2;
 - iii) Lemma 7.6.
- to the function

$$r(u) = r_{\sigma, \tau}(u) := \prod_{j=1}^N C^{u(j)} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j C^{u(j)} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^{\tau(j)},$$

we get

$$\begin{aligned}
K_{\alpha,\beta} &= \sum_{\sigma \in S_N} \det \mathcal{B}^{\alpha_\sigma, \beta_\sigma} \\
&= \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \prod_{j=1}^N (\mathcal{B}^{\alpha_\sigma, \beta_\sigma})_{j, \tau(j)} \\
&= \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \prod_{j=1}^N \operatorname{Re} \{ \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \mathcal{A}_{\beta_{\sigma(j)}, \alpha_{\sigma(j)}}^{\tau(j)} \} \\
&= \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \prod_{j=1}^N \left(\operatorname{Re} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \operatorname{Re} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^{\tau(j)} + \operatorname{Im} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \operatorname{Im} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^{\tau(j)} \right) \\
&= \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \prod_{j=1}^N \left(\sum_{u \in \{0,1\}} C^u \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \cdot C^u \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^{\tau(j)} \right) \\
&= \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \sum_{u \in \{0,1\}^N} \prod_{j=1}^N C^{u(j)} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j C^{u(j)} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^{\tau(j)} \\
&= \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \sum_{u \in \{0,1\}^N} \prod_{j=1}^N C^{u(\sigma(j))} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j C^{u(\sigma(j))} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^{\tau(j)},
\end{aligned}$$

Hence, by Remark 7.1

$$\begin{aligned}
K_{\alpha,\beta} &= \sum_{\sigma \in S_N} \det \mathcal{B}^{\alpha_\sigma, \beta_\sigma} \\
&= \sum_{u \in \{0,1\}^N} \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \prod_{j=1}^N C^{u(\sigma(j))} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \prod_{h=1}^N C^{u(\sigma(h))} \mathcal{A}_{\alpha_{\sigma(h)}, \beta_{\sigma(h)}}^{\tau(h)} \\
&= \sum_{u \in \{0,1\}^N} \sum_{\sigma \in S_N} \operatorname{sgn} \sigma \prod_{j=1}^N C^{u(\sigma(j))} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \operatorname{sgn} \sigma \sum_{\tau \in S_N} \operatorname{sgn} \tau \prod_{h=1}^N C^{u(\sigma(h))} \mathcal{A}_{\alpha_{\sigma(h)}, \beta_{\sigma(h)}}^{\tau(h)} \\
&= \sum_{u \in \{0,1\}^N} \left(\sum_{\sigma \in S_N} \operatorname{sgn} \sigma \prod_{j=1}^N C^{u(\sigma(j))} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \right) \left(\operatorname{sgn} \sigma \det \left\{ C^{u(\sigma(h))} \mathcal{A}_{\alpha_{\sigma(h)}, \beta_{\sigma(h)}}^j \right\} \right) \\
&= \sum_{u \in \{0,1\}^N} \det \left\{ C^{u(j)} \mathcal{A}_{\alpha_j, \beta_j}^h \right\} \det \left\{ C^{u(h)} \mathcal{A}_{\alpha_h, \beta_h}^j \right\} \\
&= \sum_{u \in \{0,1\}^N} \left(\det \left\{ C^{u(j)} \mathcal{A}_{\alpha_j, \beta_j}^h \right\} \right)^2 \\
&= \sum_{u \in \{0,1\}^N} \det(D(u; \alpha, \beta))^2.
\end{aligned}$$

□

Lemma 9.3. *If $\mathcal{A}^1, \dots, \mathcal{A}^N \in M_{n,sa}$ are linearly independent then there exist $\alpha, \beta \in \mathbb{C}$ and $u \in \{0,1\}^N$ such that*

$$\det \{ C^{u(j)} \mathcal{A}_{\alpha_j, \beta_j}^h \} \neq 0.$$

Proof. Note that the independence hypothesis implies $N \leq \dim_{\mathbb{R}}(M_{n,sa}) = n^2$. Therefore the $N \times n^2$ matrix

$$\begin{pmatrix} \mathcal{A}_{11}^1 & \dots & \mathcal{A}_{1n}^1 & \mathcal{A}_{21}^1 & \dots & \mathcal{A}_{2n}^1 & \dots & \mathcal{A}_{n1}^1 & \dots & \mathcal{A}_{nn}^1 \\ \mathcal{A}_{11}^2 & \dots & \mathcal{A}_{1n}^2 & \mathcal{A}_{21}^2 & \dots & \mathcal{A}_{2n}^2 & \dots & \mathcal{A}_{n1}^2 & \dots & \mathcal{A}_{nn}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathcal{A}_{11}^N & \dots & \mathcal{A}_{1n}^N & \mathcal{A}_{21}^N & \dots & \mathcal{A}_{2n}^N & \dots & \mathcal{A}_{n1}^N & \dots & \mathcal{A}_{nn}^N \end{pmatrix}$$

has rank N because it has N independent rows. This means that there exists N columns that are linearly independent and this is equivalent to say that there exists $\alpha, \beta \in \mathbb{C}$ such that the matrix

$$\begin{pmatrix} \mathcal{A}_{\alpha_1\beta_1}^1 & \mathcal{A}_{\alpha_2\beta_2}^1 & \cdots & \mathcal{A}_{\alpha_N\beta_N}^1 \\ \mathcal{A}_{\alpha_1\beta_1}^2 & \mathcal{A}_{\alpha_2\beta_2}^2 & \cdots & \mathcal{A}_{\alpha_N\beta_N}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{\alpha_1\beta_1}^N & \mathcal{A}_{\alpha_2\beta_2}^N & \cdots & \mathcal{A}_{\alpha_N\beta_N}^N \end{pmatrix}$$

has rank N . This implies that the $N \times 2N$ matrix

$$\begin{pmatrix} \operatorname{Re}\mathcal{A}_{\alpha_1\beta_1}^1 & \operatorname{Im}\mathcal{A}_{\alpha_1\beta_1}^1 & \operatorname{Re}\mathcal{A}_{\alpha_2\beta_2}^1 & \operatorname{Im}\mathcal{A}_{\alpha_2\beta_2}^1 & \cdots & \operatorname{Re}\mathcal{A}_{\alpha_N\beta_N}^1 & \operatorname{Im}\mathcal{A}_{\alpha_N\beta_N}^1 \\ \operatorname{Re}\mathcal{A}_{\alpha_1\beta_1}^2 & \operatorname{Im}\mathcal{A}_{\alpha_1\beta_1}^2 & \operatorname{Re}\mathcal{A}_{\alpha_2\beta_2}^2 & \operatorname{Im}\mathcal{A}_{\alpha_2\beta_2}^2 & \cdots & \operatorname{Re}\mathcal{A}_{\alpha_N\beta_N}^2 & \operatorname{Im}\mathcal{A}_{\alpha_N\beta_N}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \operatorname{Re}\mathcal{A}_{\alpha_1\beta_1}^N & \operatorname{Im}\mathcal{A}_{\alpha_1\beta_1}^N & \operatorname{Re}\mathcal{A}_{\alpha_2\beta_2}^N & \operatorname{Im}\mathcal{A}_{\alpha_2\beta_2}^N & \cdots & \operatorname{Re}\mathcal{A}_{\alpha_N\beta_N}^N & \operatorname{Im}\mathcal{A}_{\alpha_N\beta_N}^N \end{pmatrix}$$

has rank N because it has N independent rows. Therefore this matrix must have also N independent columns. This last assertion it is equivalent to the desired conclusion \square

Corollary 9.4. *If $(A_1)_0, \dots, (A_N)_0 \in M_{n,sa}$ are linear independent then there exist $\alpha, \beta \in \mathbb{C}$ and $u \in \{0, 1\}^N$ such that*

$$\det(D(u; \alpha, \beta)) = \det\{C^{u(j)}\mathcal{A}_{\alpha_j, \beta_j}^h\} \neq 0.$$

Proof. By definition of \mathcal{A}^j , $j = 1, \dots, N$, observe that the hypothesis of linear independence of

$$(A_1)_0, \dots, (A_N)_0$$

implies the linear independence of

$$\mathcal{A}^1, \dots, \mathcal{A}^N.$$

Hence, by Lemma 9.3 there exist $\alpha, \beta \in \mathbb{C}$ and $u \in \{0, 1\}^N$ such that

$$\det(D(u; \alpha, \beta)) = \det\{C^{u(j)}\mathcal{A}_{\alpha_j, \beta_j}^h\} \neq 0.$$

\square

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